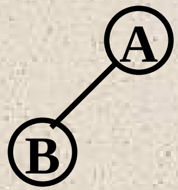
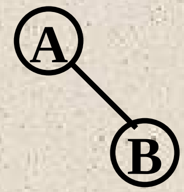
Note: **In a tree, the order of children does not matter. But in a binary tree, left child and right child are different.**

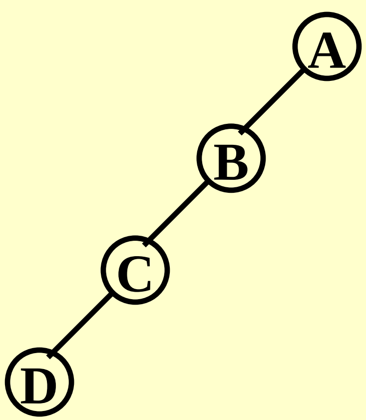


and

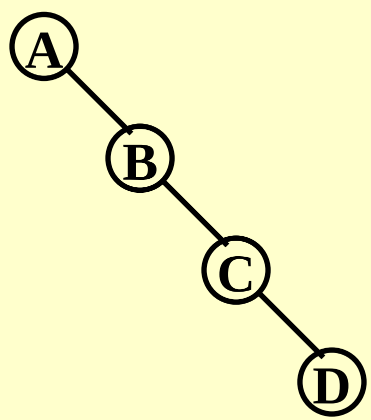


are two different binary trees.

Skewed Binary Trees

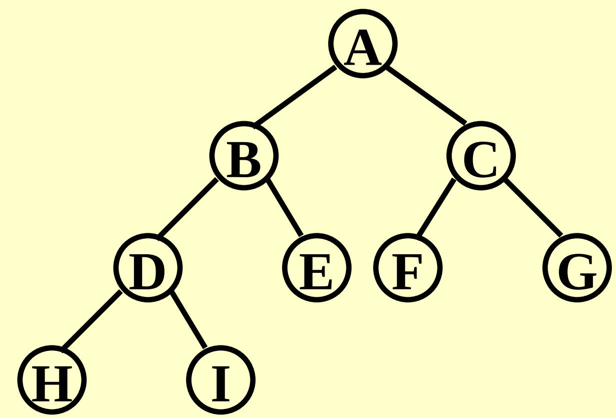


Skewed to the left



Skewed to the right

Complete Binary Tree



All the leaf nodes are on two adjacent levels

☝Properties of Binary Trees

☑ The maximum number of nodes on level i is 2^{i-1}, $i \geq 1$.
The maximum number of nodes in a binary tree of depth k is $2^k - 1$, $k \geq 1$.

☑ For any nonempty binary tree, $n_0 = n_2 + 1$ where n_0 is the number of leaf nodes and n_2 the number of nodes of degree 2.

Proof: Let n_1 be the number of nodes of degree 1, and n the total number of nodes. Then

$$n = n_0 + n_1 + n_2 \quad ①$$

Let B be the number of branches. Then $n = B + 1$. ②

Since all branches come out of nodes of degree 1 or 2, we have $B = n_1 + 2n_2$. ③

$$\Rightarrow \qquad n_0 = n_2 + 1$$

∎

§3 The Search Tree ADT -- Binary Search Trees

1. Definition

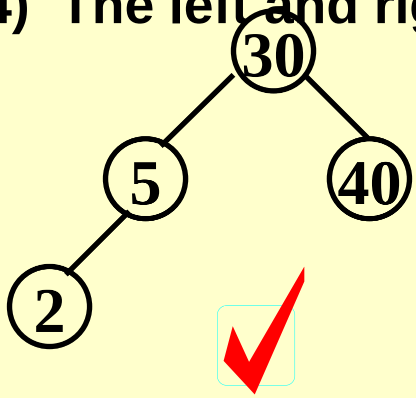
【Definition】 A binary search tree is a binary tree. It may be empty. If it is not empty, it satisfies the following properties:

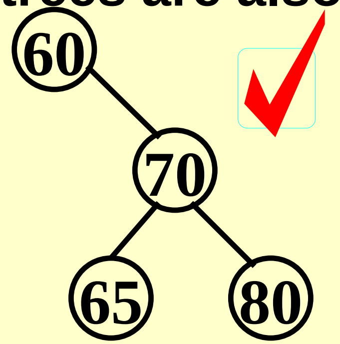
(1) Every node has a key which is an integer, and the keys are distinct.

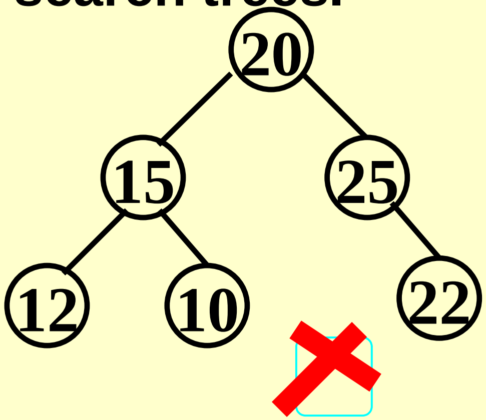
(2) The keys in a nonempty left subtree must be smaller than the key in the root of the subtree.

(3) The keys in a nonempty right subtree must be larger than the key in the root of the subtree.

(4) The left and right subtrees are also binary search trees.







2. ADT

Objects: **A finite ordered list with zero or more elements.**

Operations:

- **SearchTree MakeEmpty(SearchTree T);**
- **Position Find(ElementType X, SearchTree T);**
- **Position FindMin(SearchTree T);**
- **Position FindMax(SearchTree T);**
- **SearchTree Insert(ElementType X, SearchTree T);**
- **SearchTree Delete(ElementType X, SearchTree T);**
- **ElementType Retrieve(Position P);**

3. Implementations

Must this test be performed first?

Find

```
Position  Find( ElementType X,  SearchTree T )
{
    if ( T == NULL )
        return  NULL;  /* not found in an empty
    if ( X < T->Element )  /* if smaller than
        return  Find( X, T->Left );  /* search left
    else
        if ( X > T->Element )  /* if larger than root */
            return  Find( X, T->Right );  /* search right subtree */
        else   /* if X == root */
            return  T;  /* found */
}
```

These are tail recursions.

$T(N) = S(N) = O(d)$ **where** d **is the depth of X**

```
Position  Iter_Find( ElementType X,  SearchTree T )
{
    /* iterative version of Find */
    while  ( T )   {
        if  ( X == T->Element )
          return T ;  /* found */
        if  ( X < T->Element )
          T = T->Left ; /*move down along left path */
        else
          T = T-> Right ; /* move down along right path */
    }  /* end while-loop */
    return  NULL ;   /* not found */
}
```

FindMin

```
Position  FindMin( SearchTree T )
{
    if ( T == NULL )
       return  NULL; /* not found in an empty tree */
    else
       if ( T->Left == NULL )   return  T;  /* found left most */
       else   return  FindMin( T->Left );   /* keep moving to left */
}
```

$T(N) = O(d)$

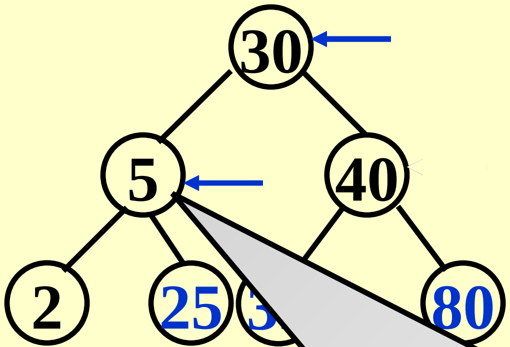
FindMax

```
Position  FindMax( SearchTree T )
{
    if ( T != NULL )
       while ( T->Right != NULL )
          T = T->Right;   /* keep moving to find right most */
    return T;  /* return NULL or the right most */
}
```

$T(N) = O(d)$

▯ Insert

Sketch of the idea:



Insert 80

① **check if 80 is already in the tree**

② **80 > 40, so it must be the right child of 40**

This is the last node we encounter when search for the key number. It will be the parent of the new node.

Insert 35

...ee

...d of 40

Insert 25

...ee

② **25 > 5, ... the right child of 5**

```
SearchTree  Insert( ElementType X, SearchTree T )
{
    if ( T == NULL ) { /* Create and return a one-node tree */
        T = malloc( sizeof( struct TreeNode ) );
        if ( T == NULL )
          FatalError( "Out of space!!!" );
        else {
          T->Element = X;
          T->Left = T->Right = NULL; }
    }  /* End creating a one-node tree */
    else  /* If there is a tree */
        if ( X < T->Element )
          T->Left = Insert( X, T->Left );
        else
          if ( X > T->Element )
            T->Right = Insert( X, T->Right );
          /* Else X is in the tree already; we'll do nothing */
   return  T;   /* Do not forget this line!! */
}
```

How would you Handle duplicated Keys?

$T(N) = O(d)$

Delete

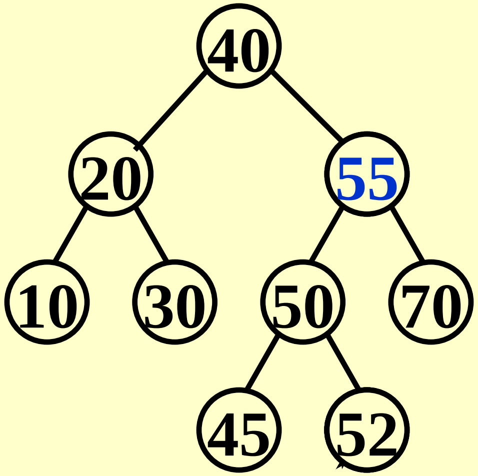
- ❖ Delete a leaf n... LL.
- ❖ Delete a degree ... its single child.
- ❖ Delete a degree 2 ... :

Note: These kinds of nodes have degree at most 1.

① Replace the node by the largest one in its left subtree or the smallest one in its right subtree.

② Delete the replacing node from the subtree.

〖Example〗 Delete 60



Solution 1: reset left subtree.

Solution 2: reset right subtree.

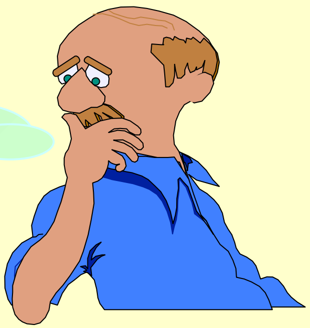
```
SearchTree  Delete( ElementType X, SearchTree T )
{    Position  TmpCell;
     if ( T == NULL )   Error( "Element not found" );
     else  if ( X < T->Element )  /* Go left */
               T->Left = Delete( X, T->Left );
          else  if ( X > T->Element )  /* Go right */
                    T->Right = Delete( X, T->Right );
               else  /* Found element to be deleted */
                 if ( T->Left && T->Right ) {  /* Two children */
                     /* Replace with smallest in right subtree */
                     TmpCell = FindMin( T->Right );
                     T->Element = TmpCell->Element;
                     T->Right = Delete( T->Element, T->Right );  } /* End if */
                 else {  /* One or zero child */
                     TmpCell = T;
                     if ( T->Left == NULL ) /* Also handles 0 child */
               T = T->Right;
                     else  if ( T->Right == NULL )  T = T->Left;
                     free( TmpCell );  }  /* End else 1 or 0 child */
     return  T;
}
```

$T(N) = O(h)$ **where h is the height of the tree**

Note:

If there are not many deletions, then *lazy deletion* may be employed: add a flag field to each node, to mark if a node is active or is deleted. Therefore we can delete a node without actually freeing the space of that node. If a deleted key is reinserted, we won't have to call malloc again.

While the number of deleted nodes is the same as the number of active nodes in the tree, will it seriously affect the efficiency of the operations?



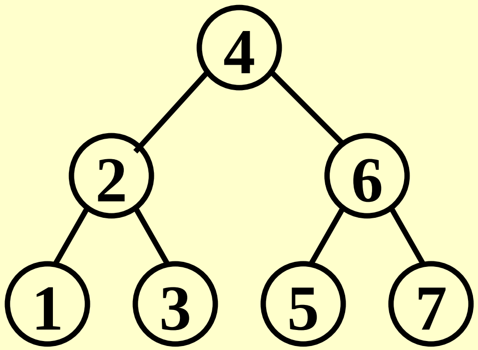
4. Average-Case Analysis

Question: Place *n* elements in a binary search tree. How high can this tree be?

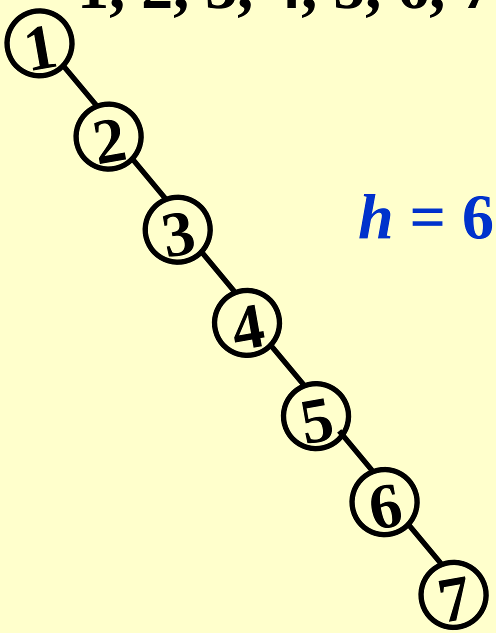
Answer: The height depends on the order of insertion.

〖Example〗 Given elements 1, 2, 3, 4, 5, 6, 7. Insert them into a binary search tree in the orders:

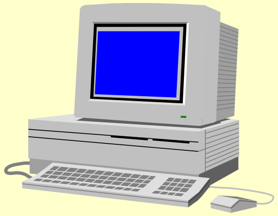
4, 2, 1, 3, 6, 5, 7 and 1, 2, 3, 4, 5, 6, 7



$h = 2$



$h = 6$



Laboratory Project 2

Normal: Tree Traversals

Hard: Voting Tree

Due: Monday, October 25th, 2021 at 10:00pm