PRIORITY QUEUES (HEAPS)

—— delete the element with the highest \ lowest priority

§1 ADT Model

Objects: **A finite ordered list with zero or more elements.**

Operations:

- PriorityQueue Initialize(int MaxElements);
- void Insert(ElementType X, PriorityQueue H);
- ElementType DeleteMin(PriorityQueue H);
- ElementType FindMin(PriorityQueue H);

§2 Simple Implementations

- **Array :**
  - **Insertion** — add one item at the end ~ $\Theta(1)$
  - **Deletion** — find the largest \ smallest key ~ $\Theta(n)$
    remove the item and shift array ~ $O(n)$
- **Linked List :**
  - **Insertion** — add to the front of the chain ~ $\Theta(1)$
  - **Deletion** — find the largest \ smallest key ~ $\Theta(n)$
    ...move the item ~ $\Theta(1)$
- **Ordered Array**
  - **Insertion** — ... ~ $O(n)$
    ... n)

Better since there are never more deletions than insertions

- **Ordered ...**
  - **Insertion** — find the proper position ~ $O(n)$
    add the item ~ $\Theta(1)$
  - **Deletion** — remove the first \ last item ~ $\Theta(1)$

Binary Search Tree :

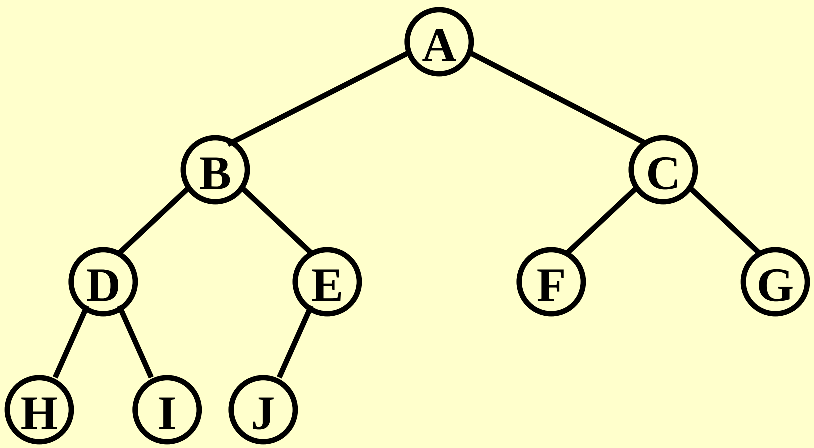
§3 Binary Heap

1. Structure Property:

【Definition】 A binary tree with n nodes and height h is **complete** iff its nodes correspond to the nodes numbered from 1 to n in the perfect binary tree of height h.

A complete binary tree of height h has between 2^h and $2^{h+1} - 1$ nodes. ➡ $h = \lfloor \log N \rfloor$

❖ Array Representation : BT [n + 1] (BT [0] is not used)



BT

0	1	2	3	4	5	6
	A	B	C	D	E	F

7	8	9	10	11	12	13
G	H	I	J			

【Lemma】 **If a complete binary tree with n nodes is represented sequentially, then for any node with index i, $1 \le i \le n$, we have:**

$$\text{(1) index of } \; parent(i) = \begin{cases} \lfloor i/2 \rfloor & \text{if } i \ne 1 \\ \text{None} & \text{if } i = 1 \end{cases}$$

$$\text{(2) index of } \; left\_child(i) = \begin{cases} 2i & \text{if } 2i \le n \\ \text{None} & \text{if } 2i > n \end{cases}$$

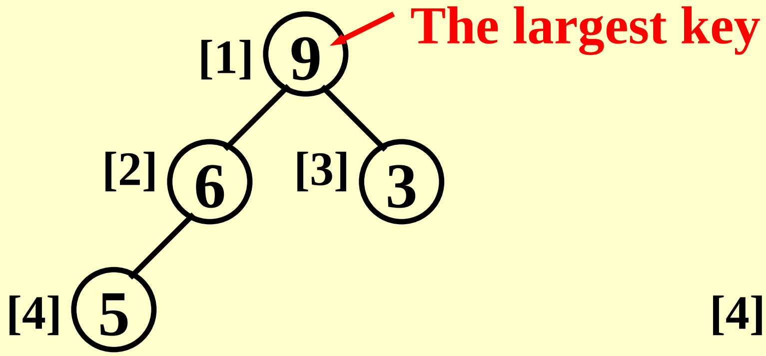
$$\text{(3) index of } \; right\_child(i) = \begin{cases} 2i+1 & \text{if } 2i+1 \le n \\ \text{None} & \text{if } 2i+1 > n \end{cases}$$

```
PriorityQueue  Initialize( int  MaxElements )
{
    PriorityQueue  H;
    if ( MaxElements < MinPQSize )
            return  Error( "Priority queue size is too small" );
    H = malloc( sizeof ( struct HeapStruct ) );
    if ( H ==NULL )
            return  FatalError( "Out of space!!!" );
    /* Allocate the array plus one extra for sentinel */
    H->Elements = malloc(( MaxElements + 1 ) * sizeof( ElementType ));
    if ( H->Elements == NULL )
            return  FatalError( "Out of space!!!" );
    H->Capacity = MaxElements;
    H->Size = 0;
    H->Elements[ 0 ] = MinData;  /* set the sentinel */
    return  H;
}
```

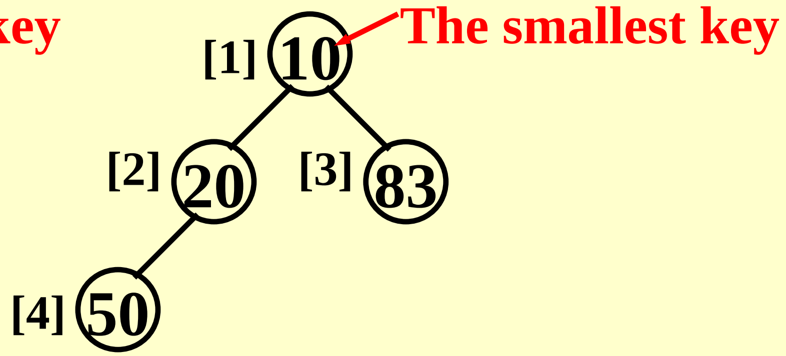
2. Heap Order Property:

【Definition】 A min tree is a tree in which the key value in each node is no larger than the key values in its children (if any). A min heap is a complete binary tree that is also a min tree.

Note: Analogously, we can declare a *max* heap by changing the heap order property.



A max heap

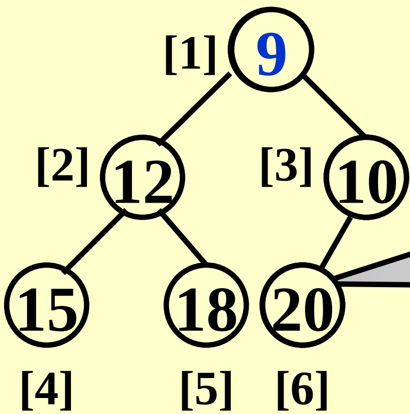


A min heap

3. Basic Heap Operations:

- **insertion**

➢ **Sketch of the idea:**



The only possible position for a new node since a heap must be a complete binary tree.

Case 1 : new_item = 21 20 < 21 ✓

Case 2 : new_item = 17 20 > 17 10 < 17 ✓

Case 3 : new_item = 9 20 > 9 10 > 9 ✓

```
/* H->Element[ 0 ] is a sentinel */
void  Insert( ElementType  X,  PriorityQueue  H )
{
    int  i;

    if ( IsFull( H ) ) {
          Error( "Priority queue is ful
          return;
    }

    for ( i = ++H->Size; H->Elements[ i / 2 ] > X; i /= 2 )
          H->Elements[ i ] = H->Elements[ i / 2 ];

    H->Elements[ i ] = X;
}
```

H->Element[0] is a *sentinel* that is no larger than the minimum element in the heap.

Faster than *swap*

$T(N) = O(\log N)$

DeleteMin

➢ **Sketch of the idea:**

Ah! That's simple -- we only have to delete the root node ...

And re-arrange the rest of the tree so that it's still a min heap.

l to keep a

ller child of 18

binary tree.

18

[4]

12 < 18

15 < 18

$T(N) = O(\log N)$

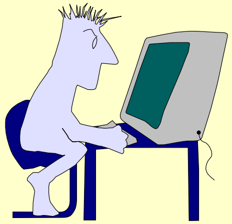
```
ElementType  DeleteMin( PriorityQueue  H )
{
    int  i, Child;
    ElementType  MinElement, LastElement;
    if ( IsEmpty( H ) ) {
        Error( "Priority queue is empty" );
        return  H->Elements[ 0 ];   }
    MinElement = H->Elements[ 1 ];  /* save the min element */
    LastElement = H->Elements[ H->Size-- ];  /* take last and reset size */
    for ( i = 1; i * 2 <= H->Size; i = Child ) {  /* Find smaller child */
        Child = i * 2;
        if (Child != H->Size && H->Elements[Child+1] < H->Elements[Child])
                Child++;
        if ( LastElement > H->Elements[ Child ] )   /* Percolate one level */
                H->Elements[ i ] = H->Elements[ Child ];
        else     break;   /* find the proper position */
    }
    H->Elements[ i ] = LastElement;
    return  MinElement;
}
```

Can we remove it by adding another sentinel?

4. Other Heap Operations:

Note: **Finding any key except the minimum one will have to take a linear scan through the entire heap.**

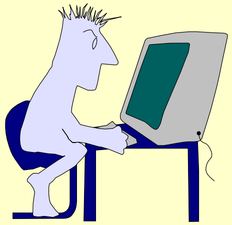
- **DecreaseKey (P, Δ, H)** *Percolate up*



sys. admin.

Lower the value of the key in the heap H at position P by a positive amount of Δ……so my programs can run with highest priority ☺.

- **IncreaseKey (P, Δ, H)** *Percolate down*



sys. admin.

Increases the value of the key in the heap H at position P by a positive amount of Δ……drop the priority of a process that is consuming excessive CPU time.

Delete (P, H)

DecreaseKey(P, ∞, H); DeleteMin(H)

Remove the node at position P from the heap H …… delete the process that is terminated (abnormally) by a user.

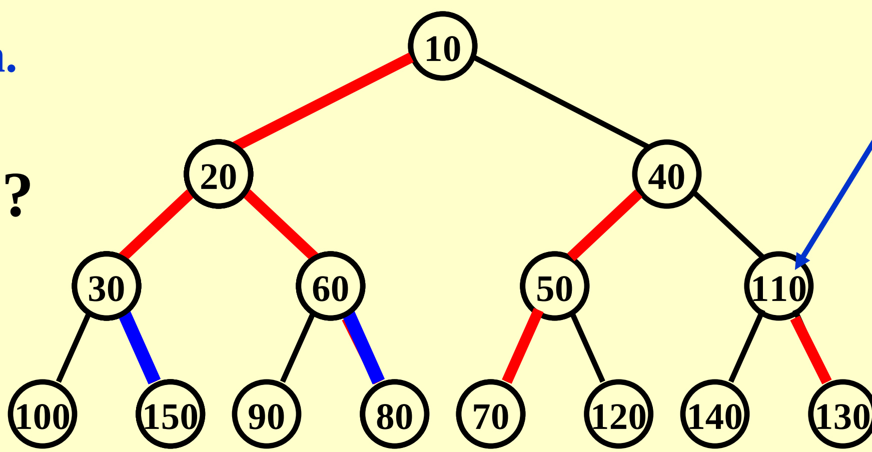
BuildHeap (H)

Nehhhhh that would be toooo slow !

Place *N* input keys into an empty heap H.

150, 80, 40, 30, 10, 70, 110, 100, 20, 90, 60, 50, 120, 140, 130

T **(*N*) = ?**



PercolateDown (7)
PercolateDown (6)
PercolateDown (5)
PercolateDown (4)
PercolateDown (3)
PercolateDown (2)
PercolateDown (1)

【Theorem】 For the perfect binary tree of height h containing $2^{h+1} - 1$ nodes, the sum of the heights of the nodes is $2^{h+1} - 1 - (h + 1)$.

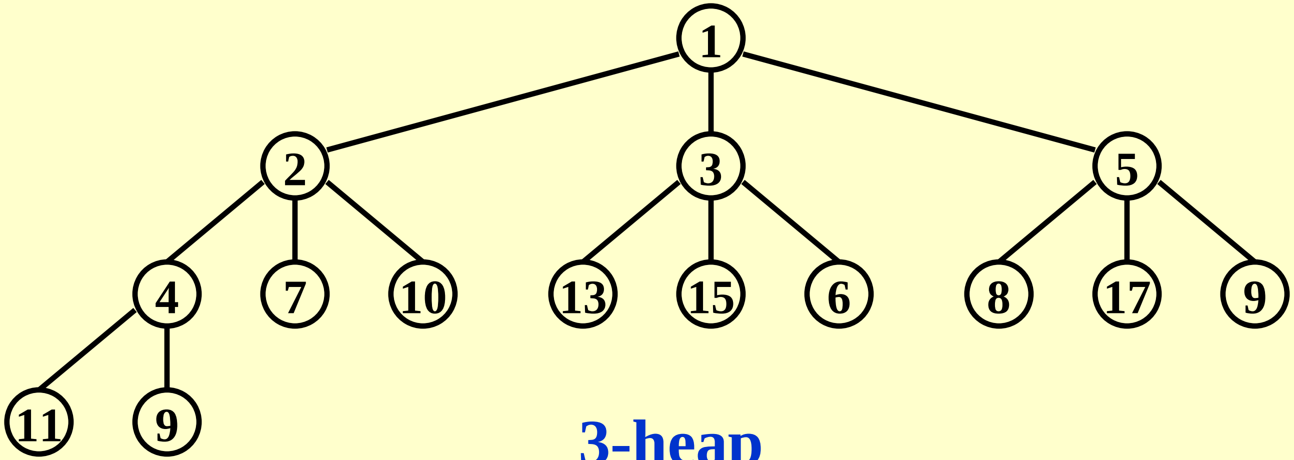
$$\longrightarrow T(N) = O(N)$$

§4 Applications of Priority Queues

〖Example〗 Given a list of N elements and an integer k. Find the kth largest element.

How many methods can you think of to solve this problem? What are their complexities?

§5 *d*-Heaps ---- All nodes have *d* children



3-heap

Question: Shall we make *d* as large as possible?

Note: ① DeleteMin will take $d - 1$ comparisons to find the smallest child. Hence the total time complexity would be $O(d \log_d N)$.

② *2 or /2 is merely a bit shift, but *d or /d is not.

③ When the priority queue is too large to fit entirely in main memory, a d-heap will become interesting.